This supplementary material hopes to accomplish the following objectives:

1. Show the general solution to the classical plate equation.
2. Demonstrate that a plate resting on an elastic foundation can be solved with the same general solution.
3. Illustrate that a plate with a finite radius requires an approximate strain energy method to solve for *a.* the displacement and *b.* the relation between load and displacement.
4. Determine the radial stress from the displacement, load, and bending moments.
5. Experimentally confirm the use of plate radius instead of characteristic length, λ, for a plate with a finite radius.

General Solution to the Equilibrium Plate Equation

The biharmonic of δ describes the classical plate equation in polar coordinates as:

$$\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\left(\frac{\partial^2\delta}{\partial r^2}+\frac{1}{r}\frac{\partial\delta}{\partial r}+\frac{1}{r^2}\frac{\partial^2\delta}{\partial\theta^2}\right)=\frac{q}{D} \quad (1a)$$

$$\nabla^4\delta=\frac{q}{D} \quad (1b)$$

where q is a uniform load on the plate and $D = Eh^3/(12(1-\nu^2)$, with E as Young's elastic modulus, h as plate thickness, and ν as Poisson's ratio. In the case of a symmetrically loaded plate (*i.e.* $\theta = 0$), the classical plate equation reduces to:

$$\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}\right)\left(\frac{d^2\delta}{dr^2}+\frac{1}{r}\frac{d\delta}{dr}\right)=\frac{q}{D} \quad (2)$$

The homogenous solution to equation 2, is given by the following series (A. Clebsch, 1862 & Timoshenko, Plates & Shells, section 62):

$$\delta = R_0+\sum_{m=1}^{\infty}R_m\cos m\theta+\sum_{m=1}^{\infty}R_m^1\sin m\theta \quad (3)$$

For a symmetrically bent plate, $\delta = R_0$, and the general solution to R_0 is:

$$\delta = R_0 = \alpha+\beta r^2+\gamma\log r+\eta_0 r^2\log r \quad (4)$$

where α, β, γ, and η are constants described by the boundary conditions. To get a finite solution at the origin, the constant γ must be equal to zero, therefore the equation describing the deflection of a circular plate resting on an elastic foundation that is loaded from the center is:

$$\delta = \alpha+\beta r^2+\eta r^2\log r \quad (5)$$

Plate on an Elastic Foundation

For the case of a circular plate on an elastic foundation we add the load $-K\delta$ to the lateral load q, where K is *modulus of the foundation* (Timoshenko, Plates & Shells, p.260):

$$\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}\right)\left(\frac{d^2\delta}{dr^2}+\frac{1}{r}\frac{d\delta}{dr}\right)=\frac{q-K\delta}{D} \quad (6)$$

In the particular case of a point load at the center (*i.e.* $q = 0$ everywhere except $r = 0$) then this equation becomes:

$$\lambda^4\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}\right)\left(\frac{d^2\delta}{dr^2}+\frac{1}{r}\frac{d\delta}{dr}\right)+\delta=0 \quad (7)$$

where $\lambda^4 = D/K$. Introducing the normalized lengths of $\delta_0 = \delta/\lambda$ and $r_0 = r/\lambda$ into equation 7, we can describe the deflection of a circular plate on an elastic foundation as:

$$\left(\frac{d^2}{dr_0^2}+\frac{1}{r_0}\frac{d}{dr_0}\right)\left(\frac{d^2\delta_0}{dr_0^2}+\frac{1}{r_0}\frac{d\delta_0}{dr_0}\right)+\delta_0=0 \quad (8a)$$

$$\nabla^4\delta_0+\delta_0=0 \quad (8b)$$

If we are interested in the deflection and stress distribution close to the center of the plate (*i.e.* $r_0 \ll \lambda$), then equation 7 can be rewritten with $r_0 = \epsilon r_1$, where ϵ is a small-order modifier of r. Timoshenko makes this approximation in Plates & Shells (section 80, towards the end). This approximation is confirmed by following his general solution to equations $(f)-(o)$ in section 57 and only considering the first order terms - *i.e.* $r \ll \lambda$.) Equation 8a becomes:

$$\left(\frac{d^2}{d(\epsilon r_1)^2}+\frac{1}{\epsilon r_1}\frac{d}{d(\epsilon r_1)}\right)\left(\frac{d^2\delta_0}{d(\epsilon r_1)^2}+\frac{1}{\epsilon r_1}\frac{d\delta_0}{d(\epsilon r_1)}\right)+\delta_0=0 \quad (9a)$$

$$\nabla^4\frac{\delta_0}{\epsilon^4}+\delta_0=0 \quad (9b)$$

By multipling the equation by ϵ to get $\nabla^4\delta_0+\epsilon^4\delta_0 = 0$ we see that in this region of interest (*i.e.* $r_0 \ll l$), the deflection of a circular plate on an elastic foundation is equal to the classical plate equation given in equation 2 because ϵ is small:

$$\nabla^4\delta_0=0 \quad (10)$$

The solution of this differential equation requires us to assume a limit of $r \ll \lambda$ (where $\lambda = (D/K)^{1/4}$). This limit implies that the pressure distribution imposed by the deflecting plate on the foundation must be nearly uniform across the entire plate radius. Accordingly, for finite circular plates with plate radii ($L \gg \lambda$), this approach is not sufficient (Timoshenko, Plates & Shells, end of section 57). For our geometry, $L \gg \lambda$; therefore, we adopt the use of an approximate strain energy method (Timoshenko, Plates & Shells, end of section 80) to determine the force-deflection relationship and associated plate stresses for a finite plate resting on a foundation, which has an effective elastic resistance to deformation ρg.

Strain Energy Method for Calculating Deflection

The strain energy method can be used to calculate the deflection of a circular plate on an elastic foundation. An approximate solution for the deflection of the circular plate on an elastic foudation (from equation 5 and Timoshenko, Plates & Shells, end of section 80):

$$\delta = \alpha+\beta r^2+\frac{P}{8\pi D}r^2\log r \quad (11)$$

where α and β are constants determined by the condition that the total energy of system in equilibrium is minimum. The term $P/(8\pi D)r^2 \log r$ describes the stress distribution around the single load at the plate's center (Timoshenko, Plates & Shells, end of section 80). The strain energy of the plate of radius L is:

$$V_1 = 4\beta^2 D\pi L^2(1+\nu) \tag{12}$$

The strain energy of the deformed elastic foundation is:

$$V_2 = \int_0^{2\pi}\int_0^L \frac{K\delta^2}{2} r dr d\theta = \pi K\left(\frac{1}{2}\alpha^2 L^2 + \frac{1}{2}\alpha\beta L^4 + \frac{1}{6}\beta^2 L^6\right) \tag{13}$$

The total energy of the system for a load P applied to the center of the plate is:

$$V = 4\beta^2 D\pi L^2(1+\nu) + \pi K\left(\frac{1}{2}\alpha^2 L^2 + \frac{1}{2}\alpha\beta L^4 + \frac{1}{6}\beta^2 L^6\right) - P\alpha \tag{14}$$

Taking the derivative this equation with respect to α and β and equating them to zero allows the constants to be determined:

$$\alpha \approx \frac{8P}{\pi K L^2} \tag{15a}$$

$$\beta \approx \frac{P}{4\pi D} \tag{15b}$$

Therefore, the deflection of a circular plate resting on an elastic foundation and loaded from the center is:

$$\delta_r = \frac{8P}{\pi K L^2} + \frac{P}{4\pi D} r^2 + \frac{P}{8\pi D} r^2 \log r \tag{16}$$

This equation leads to a relationship between load and displacement of the center of the plate. If $r = 0$, and $K = \rho g$ (from Hertz[18]), then solving equation 16 for P leads to:

$$P = \frac{\pi}{8}\rho g \delta L^2 \tag{17}$$

From these equations we can determine the radial stress distribution around the center load.

Radial Stress Distribution

When a single load, P, is applied to the center of a circular plate the radial shearing force at a distance r from the load P is:

$$Q_r = -\frac{P}{2\pi r} \tag{18}$$

The shearing force, Q_r, is equal to the partial derivative of the moment M_r with respect to r. The maximum bending moment is at the center of the plate and is given by:

$$M_r = -D\left[\frac{\partial^2\delta}{\partial r^2} + \nu\left(\frac{1}{r}\frac{\partial\delta}{\partial r} + \frac{1}{r^2}\frac{\partial^2\delta}{\partial\theta^2}\right)\right] \tag{19}$$

The corresponding maximum radial stress is (Timoshenko, Plates & Shells, section 16):

$$\sigma_{rr} = \frac{6M_r}{h^2} \tag{20}$$

The bending moment in equation 19 is determined from equation 16, which leads to:

$$M_r = -D\left(\frac{7P}{8\pi D} + \frac{P}{4\pi D}\log r + \nu\frac{5P}{8\pi D} + \nu\frac{P}{4\pi D}\log r\right) \tag{21}$$

Therefore, the radial stress becomes:

$$\sigma_{rr} = \frac{-6P}{\pi h^2}\left(\frac{7}{8} + \frac{5}{8}\nu + \frac{1}{4}\log r\right) \tag{22}$$

Using equation 17 (which assumes $r = 0$) along with $\nu = 0.3$ leads to an approximate radial stress:

$$\sigma_{rr} \approx \rho g \delta \left(\frac{L}{h}\right)^2 \tag{23}$$

Confirmation of L for a Finite Radius Plate

The strain energy method for determining the displacement of a circular plate resting on an elastic foundation that is loaded from the center is necessary for a plate with a finite radius that is greater than the characteristic length λ. To confirm experimentally that plate radius, L, is the appropriate length scale for understanding the relationship between force and displacement we plot P vs. δ for a wide range of film thicknesses in figure 1. Equation 17 indicates that the slope of a force versus displacement curve should be independent of film thickness and dependent on the radius of the film. The P vs δ curves have a slope corresponding to $L = 20.5mm$, an error of 17% from our experimentally measured radius, $L = 17.5mm$, thus confirming the relation over a large range of h.

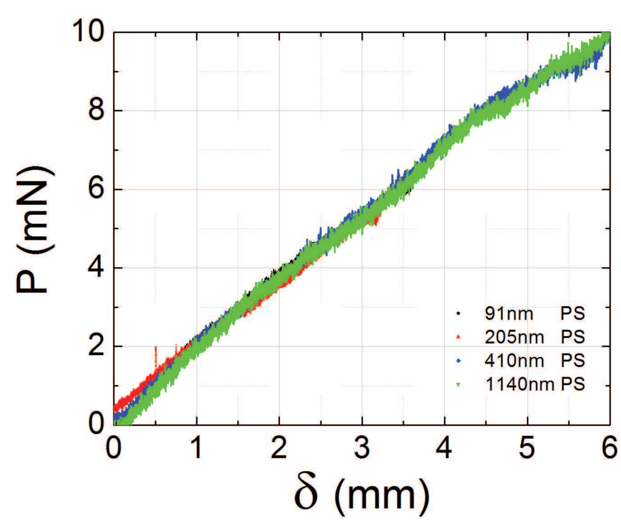

Figure 1: Plot of force, P, vs. displacement, δ, for PS films over a wide range of film thickness, h.